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A scalar invariant and the local geometry of a class of static spacetimes

Manash Mukherjee

Department of Mathematics, University of California, Davis, CA 95616
E-mail: manash@math.ucdavis.edu

F. Paul Esposito

Department of Physics, University of Cincinnati, Cincinnati, OH 45221
E-mail: esposito@physics.uc.edu

L. C. R. Wijewardhana

Department of Physics, University of Cincinnati, Cincinnati, OH 45221 and Institute of Fundamental Studies, Kandy, Sri Lanka
E-mail: rohana@physics.uc.edu

Abstract: The scalar invariant, $I \equiv R_{\mu\nu\rho\sigma;\delta}\ R^{\mu\nu\rho\sigma;\delta}$, constructed from the covariant derivative of the curvature tensor is used to probe the local geometry of static spacetimes which are also Einstein spaces. We obtain an explicit form of this invariant, exploiting the local warp-product structure of a 4-dimensional static spacetime, ${}^{(3)}\Sigma \times_f \mathbb{R}$, where ${}^{(3)}\Sigma$ is the Riemannian hypersurface orthogonal to a timelike Killing vector field with norm given by a positive function, $f :\ {}^{(3)}\Sigma \longrightarrow \mathbb{R}$. For a static spacetime which is an Einstein space, it is shown that the locally measurable scalar, I, contains a term which vanishes if and only if ${}^{(3)}\Sigma$ is conformally flat; also, the vanishing of this term implies (a) ${}^{(3)}\Sigma$ is locally foliated by level surfaces of f, ${}^{(2)}S$, which are totally umbilic spaces of constant curvature, and (b) ${}^{(3)}\Sigma$ is locally a warp-product space, $\mathbb{R}\times_{\ r(f)}\ {}^{(2)}S$, for some function $r(f)$. Futhermore, if ${}^{(3)}\Sigma$ is conformally flat it follows that every non-trivial static solution of the vacuum Einstein equation with a cosmological constant, is either Nariai-type or Kottler-type - the classes of spacetimes relevant to quantum aspects of gravity.

Keywords: Classical Theories of Gravity, Differential and Algebraic Geometry.

Contents

1. Introduction

It is well known from the work of Cartan [1] and Thomas [2] that the local geometry of a semi-Riemannian manifold is uniquely determined up to isometries by the curvature tensor and its covariant derivatives up to a finite order. This result, as suggested by Brans [3] and further developed by Karlhede [4], may be used to provide a coordinate independent characterization of gravitational fields in general relativity. Based on this idea, Karlhede et al [5] have investigated the simplest scalar invariant, $I \equiv R_{\mu\nu\rho\sigma;\delta}\ R^{\mu\nu\rho\sigma;\delta}$, constructed from the first covariant derivative of the curvature tensor, and found that the behavior of this locally measurable scalar reveals the effect of passage through the event horizon of the Schwarzschild spacetime - a static solution to the vacuum Einstein equation, $R_{\mu\nu} = 0$. For any static spacetime, however, we observe that I contains a term I_B which is the "square" of the rank-3 tensor $B \equiv \nabla_V R(V)|_\Sigma$ on the hypersurface Σ orthogonal to the timelike Killing vector field V. In particular, I_B vanishes for the Schwarzschild spacetime where Σ is known to be conformally flat. Interestingly, we find that the tensor $\nabla_V R(V)|_\Sigma$ is proportional to the Cotton tensor [6] on the spacelike hypersurface Σ, for *every* static solution to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ - the vacuum Einstein equation with a cosmological constant Λ. It is well-known that a semi-Riemannian 3-manifold is conformally flat if and only if the Cotton tensor vanishes. Hence, for all static solutions to $R_{\mu\nu} = \Lambda g_{\mu\nu}$, Σ is conformally flat if and only if $I_B = 0$. The principal goal of this paper is to determine all non-trivial *static spacetimes which are also Einstein spaces* (henceforth, referred to as "STE" spacetimes), satisfying $I_B = 0$.

In fact, I_B vanishes for a rich variety of STE spacetimes (that is, static solutions to $R_{\mu\nu} = \Lambda g_{\mu\nu}$): (*a*) with $\Lambda = 0$, the Levi-Civita solutions in Class A [7] which contains the Schwarzschild spacetime, (*b*) with $\Lambda > 0$, the Nariai [8] solution and (*c*) with arbitrary Λ, the Kottler [9] solution that includes the solutions in (*a*), de Sitter space (dS, $\Lambda > 0$), anti-de Sitter space (AdS, $\Lambda < 0$) and Schwarzschild-dS (-AdS) spacetimes [11]. The Nariai solution is important for its quantum instability: once created, it decays into a pair of near extreme black holes [10]. Also, generalizing the Kottler solution, there exist *topological* black holes asymptotic to anti-de Sitter space where the horizon can be a Riemann surface with arbitrary genus [12]. The topological black holes are relevant for testing (1) the so called "AdS/CFT correspondence" [13] in higher spacetime dimensions and (2) the Horowitz-Polchinski [14] correspondence principle which provides microscopic interpretation of black hole entropy by linking a highly excited string state to a black hole (see also [15]). It has been demonstrated [16] that topological black holes of arbitrary genus could result from the gravitational collapse of pressureless dust. Thus the local geometry of STE spacetimes with $I_B = 0$ is of considerable interest. In general, for the class of 4-dimensional STE spacetimes, we find that the vanishing of I_B implies that ${}^{(3)}\Sigma$ is locally foliated by 2-dimensional surfaces, ${}^{(2)}S$, of constant curvature. From the resulting structure of ${}^{(3)}\Sigma$, we show that *any* nonflat static solution to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ must be either Kottler-type or Nariai-type [subsection 3.4] *if* $I_B = 0$.

To fix our notations, we recall that there is a coordinate system $\{x^0,\ x^1,\ x^2,\ x^3\}$ at each point of a static spacetime, $(M^4,\ g)$, with $g_{0i} = 0$ and $\partial_0 g_{\mu\nu} = 0$ for $i = 1,\ 2,\ 3$ and $\mu,\ \nu = 0, 1,\ 2,\ 3$. Here ∂_0 is the timelike, twist-free Killing vector field on $(M^4,\ g)$. Thus, $(M^4,\ g)$ may be locally represented as a warp-product [17],

$$ {}^{(3)}\Sigma \times_f \mathbb{R} \quad \text{with} \tag{1.1} $$

$$ g = h_{ij} dx^i \otimes dx^j + f^2(-dx^0 \otimes dx^0)\ , \tag{1.2} $$

where $h_{ij} = g_{ij}$ is the Riemannian metric on the hypersurface ${}^{(3)}\Sigma$, and the function f on ${}^{(3)}\Sigma$ is defined by, $f = \sqrt{-g_{00}} > 0$. Throughout the paper, lower case Latin indices will represent "spatial" coordinates on ${}^{(3)}\Sigma$ and the Greek indices will denote spacetime coordinates. The curvature tensor, Ricci tensor and scalar curvature on ${}^{(3)}\Sigma$ will be denoted by $\hat{R}_{ijkl}$, $\hat{R}_{ij}$ and $\hat{R}$, respectively. On ${}^{(3)}\Sigma$, we also define the Hessian of f by the symmetric rank-2 tensor, $H(\partial_i,\ \partial_j) \equiv h\left(\hat{\nabla}_{\partial_i}\widetilde{df},\ \partial_j\right) \equiv H_{ij}$, where $\widetilde{df} \equiv (\partial^k f)\partial_k \equiv f^k \partial_k$ is the gradient of f, and $\hat{\nabla}$ is the Levi-Civita connection on $(\ {}^{(3)}\Sigma,\ h)$.

The paper is organized as follows. In section 2 we obtain an explicit formula for the invariant, $I \equiv R_{\mu\nu\rho\sigma;\delta}\ R^{\mu\nu\rho\sigma;\delta}$, on a static spacetime and briefly outline its local measurability. Here, the distinguished components, $B_{ijk} \equiv R_{0ijk;0}$, of the covariant derivative of curvature define a rank-3 tensor on $(\ {}^{(3)}\Sigma,\ h)$, and the scalar $I_B \equiv R_{0ijk;0}R^{0ijk;0}$ vanishes if and only if $R_{0ijk;0} = 0$ since h is Riemannian. If a static spacetime is also an Einstein space, we prove in Proposition 1 that $R_{0ijk;0} = (-f^2)\hat{R}_{ijk}$, where $\hat{R}_{ijk}$ is the Cotton tensor [6] on the Riemannian 3-manifold, $(\ {}^{(3)}\Sigma,\ h)$.

In section 3 we consider a domain of (${}^{(3)}\Sigma,\ h$) where $||\widetilde{df}||^2 \equiv h(\widetilde{df}, \widetilde{df}) \neq 0$ and determine the local geometry of STE spacetimes subjected to the condition, $R_{0ijk;0} = 0$. First, we derive the characteristic structure of the Hessian of f, H_{ij}, which specifies the Weyl curvature operator for an STE spacetime. While any static spacetime is known to be either type I, D or O, we prove that a static spacetime which is also an Einstein space must be locally of type D or O if $R_{0ijk;0} = 0$. Furthermore, in the domain where $||\widetilde{df}|| \neq 0$, the 1-form df defines a local foliation of (${}^{(3)}\Sigma,\ h$) and we determine the extrinsic and intrinsic geometry of the integral manifolds, ${}^{(2)}S$. From the local structure of ${}^{(3)}\Sigma$, we derive all non-trivial STE spacetimes satisfying $R_{0ijk;0} = 0$. Finally, in section 4 we discuss possible generalizations of our main results which can be summarized as follows.
Main Theorem: Let ${}^{(3)}\Sigma \times_f \mathbb{R}$ be a static solution to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ (STE spacetime). (1) ${}^{(3)}\Sigma$ is conformally flat if and only if $R_{0ijk;0} = 0$. (2) The spacetime is locally of type D or O if $R_{0ijk;0} = 0$. (3) ${}^{(3)}\Sigma$ is locally foliated by totally umbilic 2-manifolds, ${}^{(2)}S$, of constant curvature if $R_{0ijk;0} = 0$ and hence, (${}^{(3)}\Sigma,\ h$) is locally a warp-product space, $\mathbb{R}\times\ _{r(f)}\ {}^{(2)}S$, for some function $r(f)$: if $r(f)$ is a constant, a non-trivial STE spacetime is Nariai-type; otherwise it is Kottler-type.

2. Invariant, I, for static solutions (${}^{(3)}\Sigma \times_f \mathbb{R}$) to $R_{\mu\nu} = \Lambda g_{\mu\nu}$

2.1 Curvature components and explicit form of I

For a static spacetime, M^4 locally represented by ${}^{(3)}\Sigma \times_f \mathbb{R}$, the nonzero covariant derivatives are

$$\nabla_{\partial_i}\partial_j = \hat{\Gamma}^k_{ij}\partial_k, \tag{2.1}$$

$$\nabla_{\partial_i}\partial_0 = \left(\frac{\partial_i f}{f}\right)\partial_0\ , \quad \text{and} \tag{2.2}$$

$$\nabla_{\partial_0}\partial_0 = f\left(\partial^i f\right)\partial_i\ . \tag{2.3}$$

Remark 1 *The functions, $\hat{\Gamma}^k_{ij}$, in (2.1) are the Christoffel components on (${}^{(3)}\Sigma, h$). Thus, the second fundamental form of ${}^{(3)}\Sigma$ vanishes identically, and hence, $\nabla_{\partial_i}\partial_j = \hat{\nabla}_{\partial_i}\partial_j$, where $\hat{\nabla}$ is the Levi-Civita connection on (${}^{(3)}\Sigma,\ h$).*

Using (2.1) - (2.3), components of the curvature tensor for any static spacetime M^4 can be expressed in terms of the data on ${}^{(3)}\Sigma$:

$$R_{0ijk} = 0\ , \tag{2.4}$$

$$R_{0i0j} = fH_{ij}\ , \quad \text{and} \tag{2.5}$$

$$R_{ijkl} = \hat{R}_{ijkl}\ , \tag{2.6}$$

where the symmetric tensor field, $H_{ij} \equiv H(\partial_i,\ \partial_j)$ represents the Hessian of the function $f(x^1,\ x^2,\ x^3)$ on (${}^{(3)}\Sigma,\ h$),

$$H(\partial_i,\ \partial_j) = \partial_i\partial_j f - \hat{\Gamma}^k_{ij}\partial_k f. \tag{2.7}$$

Remark 2 *Since the extrinsic curvature of* $({}^{(3)}\Sigma,\ h)$ *is zero, the "spatial" components of the curvature tensor for the spacetime* $(M^4,\ g)$ *are given by* $R_{ijkl} = \hat{R}_{ijkl}$ *, where* $\hat{R}_{ijkl}$ *is described below.*

The curvature tensor, $\hat{R}_{ijkl}$, of $({}^{(3)}\Sigma,\ h)$, is given by

$$\begin{aligned}\hat{R}_{ijkl} &= \frac{\hat{R}}{2}\{h_{il}h_{jk} - h_{ik}h_{jl}\} \\ &+ \left\{\hat{R}_{ik}h_{jl} - \hat{R}_{il}h_{jk}\right\} + \left\{\hat{R}_{jl}h_{ik} - \hat{R}_{jk}h_{il}\right\} \ , \end{aligned} \tag{2.8}$$

where the Ricci tensor and scalar curvature of $({}^{(3)}\Sigma,\ h)$ are related to the Ricci tensor, $R_{\mu\nu}$, of $(M^4,\ g)$ by

$$\hat{R}_{ij} = \frac{1}{f}H_{ij} + R_{ij} \quad \text{and} \tag{2.9}$$

$$\hat{R} \equiv h^{ij}\hat{R}_{ij} = \frac{R_{00}}{f^2} + h^{ij}R_{ij} \ . \tag{2.10}$$

For a static spacetime, $(M^4,\ g)$, the scalar invariant, $I \equiv R_{\mu\nu\rho\sigma;\delta}R^{\mu\nu\rho\sigma;\delta}$, is given by

$$I = h^{mn}R_{ijkl;m}R^{ijkl}{}_{;n} + 4h^{mn}R_{0i0j;m}R^{0i0j}{}_{;n} + 4(g^{00})^2R_{0ijk;0}R_0{}^{ijk}{}_{;0} \tag{2.11}$$

The quantities appearing in the above equation are derived from locally measurable components of the covariant derivative of the curvature tensor [5]. Using normal coordinates ξ^A in a field of frames parallel transported along the geodesics from $\xi^A = 0$, the curvature tensor can be expanded as [18]

$$R_{ABCD}(\xi) = R_{ABCD}(0) + R_{ABCD;E}(0)\xi^E + O(\xi^2).$$

From the equation for geodesic deviation, one finds the curvature components by measuring [19] relative accelerations of test particles moving with various speed and directions in the neighborhood of $\xi^A = 0$. Hence, the covariant derivatives of the curvature tensor at $\xi^A = 0$ can be obtained from the above expansion.
If a static spacetime, (M^4, g), is also an Einstein space,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{2.12}$$

for some constant $\Lambda \in \mathbb{R}$, then we have $R_{ij} = \Lambda h_{ij}$, $R_{0j} = 0$ and $R_{00} = -\Lambda f^2$. Also, from the equations (2.9) and (2.10), the Ricci tensor and scalar curvature on $({}^{(3)}\Sigma,\ h)$ are given by

$$\hat{R}_{ij} = \frac{1}{f}H_{ij} + \Lambda h_{ij}, \tag{2.13}$$

$$\hat{R} = h^{ij}\hat{R}_{ij} = \frac{1}{f}(trH) + h^{ij}R_{ij} = 2\Lambda, \tag{2.14}$$

where $\frac{1}{f}(trH) = -\Lambda$. Inserting (2.4)-(2.8) and (2.13)-(2.14) in (2.11) we have

$$I = 8\ \hat{R}_{ij;k}\hat{R}^{ij;k} + \frac{4}{f^4}R_{0ijk;0}R_0{}^{ijk}{}_{;0}. \tag{2.15}$$

2.2 Probing conformal flatness of ${}^{(3)}\Sigma$

As noted in section 1, the significance of the second term in the scalar invariant, I, for the local geometry of an STE spacetime is derived from the relation between $R_{0ijk;0}$ and the Cotton tensor. This relation is established in the following proposition.

Proposition 1: In a static spacetime $({}^{(3)}\Sigma \times_f \mathbb{R})$, $R_{0ijk;0}$ defines a rank-3 tensor field on $({}^{(3)}\Sigma,\ h)$. If $({}^{(3)}\Sigma \times_f \mathbb{R})$ is also an Einstein space then $R_{0ijk;0}$ is related to the Cotton tensor, $\hat{R}_{ijk}$, by

$$R_{0ijk;0} = (-f^2)\hat{R}_{ijk}. \tag{2.16}$$

Proof. The contracted Bianchi identity [20], $R^{\mu}_{\nu\rho\sigma;\mu} = R_{\nu\sigma;\rho} - R_{\nu\rho;\sigma}$, implies that

$$R^{0}_{ijk;0} + R^{m}_{ijk;m} = R_{ik;j} - R_{ij;k}.$$

Using (2.6) and (2.9) in the above equation, and inserting (2.13)-(2.14) in the definition of the Cotton tensor, $\hat{R}_{ijk} = (\hat{R}_{ij;k} - \hat{R}_{ik;j}) - \frac{1}{4}(h_{ij}\hat{R}_{;k} - h_{ik}\hat{R}_{;j})$, we have

$$\left(-\frac{1}{f^2}\right) R_{0ijk;0} = \left(\frac{1}{f}H_{ij}\right)_{;k} - \left(\frac{1}{f}H_{ik}\right)_{;j} = \hat{R}_{ijk}. \tag{2.17}$$

Now, Proposition 1 follows from the equation (2.17), where the first equality holds for any static spacetime.

Using (2.16) in (2.15), the scalar invariant for an STE spacetime is now given by

$$I = 8\ \hat{R}_{ij;k}\hat{R}^{ij;k} + 4\ \hat{R}_{ijk}\hat{R}^{ijk}\ . \tag{2.18}$$

Thus, in I [(2.15)], vanishing of the term, $I_B \equiv R_{0ijk;0}R^{0ijk;0} = \hat{R}_{ijk}\hat{R}^{ijk}$, is equivalent to conformal flatness of $(\ {}^{(3)}\Sigma,\ h)$ since a 3-dimensional Riemannian manifold is conformally flat if and only if the Cotton tensor, $\hat{R}_{ijk} = 0$ [6]. This proves the part-(1) of the Main Theorem as stated at the end of section 1.

For the Schwarzschild spacetime with $\Lambda = 0$, $f^2 = (1 - r_s/r)$ and $h = (dr \otimes dr)/f^2 + r^2(d\theta \otimes d\theta + sin^2\theta\ \ d\phi \otimes d\phi)$, the term $I_B \equiv R_{0ijk;0}R^{0ijk;0}$ in I vanishes, and hence,

$$I = 8\hat{R}_{ij;m}\hat{R}^{ij;m} = 180\frac{r_s^2}{r^8}\left(1 - \frac{r_s}{r}\right),$$

where r_s the Schwarzschild radius. Hence, the local measurement of the scalar invariant, I, not only reveals the passage through the event horizon [5] but also the conformal flatness of the spacelike hypersurface, $(\ {}^{(3)}\Sigma,\ h)$. In the next section we turn to further consequences of $I_B \equiv R_{0ijk;0}R^{0ijk;0} = 0$ for the local geometry of STE spacetimes.

3. Conformally flat ${}^{(3)}\Sigma$ and static solutions to $R_{\mu\nu} = \Lambda g_{\mu\nu}$

To prove the part-(2) and part-(3) of the Main Theorem, we obtain an explicit form of $R_{0ijk;0}$ for an STE spacetime. For a static spacetime, it follows from the first equality in (2.17) that

$$R_{0ijk;0} = f_k H_{ij} - f_j H_{ik} - f(H_{ij;\ k} - H_{ik;\ j}), \tag{3.1}$$

where $f_i \equiv \partial_i f$. Since $H_{ij} = \partial_i\partial_j f - \hat{\Gamma}^k_{ij}\partial_k f = f_{i;j}$, the Ricci identity for the covariant vector field, f_i, on $({}^{(3)}\Sigma,\ h)$, gives

$$H_{ij;k} - H_{ik;j} \ = \ f_{i;jk} - f_{i;kj} = f^m \hat{R}_{mijk}.$$

Inserting the above identity in (3.1), we find

$$R_{0ijk;0} \ = \ H_{ij} f_k - H_{ik} f_j - f f^m \hat{R}_{mijk}. \tag{3.2}$$

Replacing $\hat{R}_{mijk}$ in (3.2) by (2.8) and using (2.13)-(2.14) we have

$$R_{0ijk;0} = 2(H_{ij}f_k - H_{ik}f_j) - h_{ij}\{(trH)f_k - H_{km}f^m\} + h_{ik}\{(trH)f_j - H_{jm}f^m\}, \tag{3.3}$$

in STE spacetimes. Here and in the rest of the paper, we consider a domain of $(\ {}^{(3)}\Sigma,\ h)$ where $||\widetilde{df}||^2 \equiv h(\widetilde{df}, \widetilde{df}) \neq 0$ and hence, the 1-form df defines a local foliation of ${}^{(3)}\Sigma$. Integral manifolds of this foliation are spacelike hypersurfaces, ${}^{(2)}S$, which are, locally, $f = constant$ surfaces in $(\ {}^{(3)}\Sigma,\ h)$. The local vector field $N_k \equiv f_k/||\widetilde{df}||$ is the unit normal, and the symmetric tensor field, $\bar{h}_{ij} = h_{ij} - N_i N_j$, is the induced metric on ${}^{(2)}S \subset\ {}^{(3)}\Sigma$.

3.1 Structure of the Hessian of f

First, we prove a basic result that $R_{0ijk;0} = 0$ (or conformal flatness of ${}^{(3)}\Sigma$) imprints a characteristic structure on the Hessian H_{ij} in STE spacetimes: H_{ij} has an eigenvalue $H(N,N)$ with the 1-dimensional eigenspace along N, and a repeated eigenvalue $\{(trH) - H(N,N)\}/2$ with the tangent plane to ${}^{(2)}S$ as the 2-dimensional eigenspace, at each point of the regular domain of ${}^{(3)}\Sigma$.

Proposition 2: In a domain of an STE spacetime $({}^{(3)}\Sigma \times_f \mathbb{R})$ where $df \neq 0$, the rank-3 tensor, $R_{0ijk;0}$, on $(\ {}^{(3)}\Sigma,\ h)$ vanishes *if and only if*

$$H_{ij} = \frac{1}{2}\{(trH) - H(N,N)\}\bar{h}_{ij} + H(N,N)N_i N_j, \tag{3.4}$$

where $N = \widetilde{df}/||\widetilde{df}||$ is the unit normal field on the hypersuface, $(\ {}^{(2)}S,\ \bar{h}) \subset (\ {}^{(3)}\Sigma,\ h)$.

Proof. Setting $R_{0ijk;0} = 0$ and $f_k = (||\widetilde{df}||)N_k$ in (3.3), we have

$$2(H_{ij}N_k - H_{ik}N_j) - h_{ij}\{(trH)N_k - H_{km}N^m\} + h_{ik}\{(trH)N_j - H_{jm}N^m\} = 0. \tag{3.5}$$

Contracting both sides of (3.5) by $N^i N^k$, we find

$$H_{ij}N^i = H(N,N)N_j. \tag{3.6}$$

Now, inserting (3.6) back in (3.5), leads to

$$\{2H_{ij} - (trH - H(N,N))h_{ij}\}N_k - \{2H_{ik} - (trH - H(N,N))h_{ik}\}N_j = 0. \tag{3.7}$$

Finally, contracting (3.7) by N^k, we have (3.4). Conversely, if the equation (3.4) holds, then contracting with N^i, we get (3.6). Now, inserting (3.4) and (3.6) in (3.3), we find $R_{0ijk;0} = 0$.

3.2 Petrov types

Corollary: For an STE spacetime, $R_{0ijk;0} = 0$ implies that the spacetime is locally of type D or O.

Proof. We choose an oriented orthonormal basis $\{e_0 = \partial_0/f,\ e_1 = N,\ e_2,\ e_3\}$ for the tangent space at each point of the spacetime. Using this basis, the matrix of the Weyl curvature operator - a self adjoint linear transformation

$$\mathbf{C} : \Lambda^2_* \longrightarrow \Lambda^2_* \tag{3.8}$$

on the 3-dimensional complex vector space of bivectors (Λ^2_*) - is given by [21]

$$[\mathbf{C}] = -A + iB, \tag{3.9}$$

where 3×3 real symmetric traceless matrices A and B have the following form:

$$A_{0i,0j} = C(e_0 \wedge e_i,\ e_0 \wedge e_j) \equiv C(e_0,\ e_i,\ e_0,\ e_j), \tag{3.10}$$

$$B_{0i,jk} = C(e_0 \wedge e_i,\ e_j \wedge e_k) \equiv C(e_0,\ e_i,\ e_j,\ e_k), \tag{3.11}$$

for $i,\ j,\ k = 1, 2, 3$. Here, a double index "$\mu\nu$" represents a basis element, $e_\mu \wedge e_\nu$, of the real bivector space, Λ^2; the Weyl cuvature tensor, C, induces a symmetric bilinear form (also denoted by C) on Λ^2,

$$C : \Lambda^2 \times \Lambda^2 \longrightarrow \mathbb{R}, \tag{3.12}$$

defined by $C(e_\mu \wedge e_\nu,\ e_\rho \wedge e_\sigma) \equiv C(e_\mu,\ e_\nu,\ e_\rho,\ e_\sigma)$; $A_{0i,0j}$ and $B_{0i,jk}$ are the distinguished matrix elements of this bilinear form (see [21] for the details). For an STE spacetime, the equations (3.10) and (3.11), give

$$A_{0i,0j} = \frac{1}{f}\left\{H(e_i, e_j) - \left(\frac{trH}{3}\right) g(e_i, e_j)\right\}, \tag{3.13}$$

$$B_{0i,jk} = 0. \tag{3.14}$$

If $R_{0ijk;0} = 0$, it follows from the equation (3.4) in Proposition 2 that

$$H(e_i, e_j) = 0, \quad \text{for } i \neq j \tag{3.15}$$

$$= \frac{1}{2}\{trH - H(e_1, e_1)\} \quad \text{for } i = j = 2,\ 3. \tag{3.16}$$

Now, the matrix of the Weyl curvature operator can be obtained from the equations (3.9) and (3.13)-(3.16),

$$[\mathbf{C}] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where $\lambda_2 = \lambda_3 = -\lambda_1/2$ and $\lambda_1 = -\{H(N, N) - (trH)/3\}/f$. Hence, the spacetime is locally of type D if $\lambda_1 \neq 0$, and type O if $\lambda_1 = 0$. This proves part-(2) of the Main Theorem.

Remark 3 *Our solution for H_{ij} in (3.4) [*Proposition 2*] can be written as*

$$H_{ij} = \frac{1}{2}\{(trH) - H(N,N)\}h_{ij} + \frac{3}{2}\{H(N,N) - (trH)/3\}N_i N_j,$$

where we used $\bar{h}_{ij} = h_{ij} - N_i N_j$. Hence, the solution, $H_{ij} = \{(trH) - H(N,N)\}h_{ij}/2$, is a special case of our solution in (3.4) when $\{H(N,N) - (trH)/3\} = 0$. In fact, this special case corresponds to $\lambda_1 = 0$ that leads to a locally type O spacetime.

3.3 Local warp-product structure of ${}^{(3)}\Sigma$

To complete the proof of the Main Theorem, we need to characterise the extrinsic geometry of ${}^{(2)}S \subset {}^{(3)}\Sigma$ in STE spacetimes when $R_{0ijk;0} = 0$. Using the projection tensor, $\bar{h}^i_j = \delta^i_j - N^i N_j$, we introduce the extrinsic curvature [20] of ${}^{(2)}S$,

$$K_{ij} \equiv \bar{h}^m_i \bar{h}^k_j N_{k;m} = \frac{1}{||\widetilde{df}||} \bar{h}^m_i \bar{h}^k_j H_{mk}. \tag{3.17}$$

Now, we observe that the equation (3.4) holds if and only if

$$\begin{aligned} H_{il} N^i \bar{h}^l_m &= 0 \quad \text{and} \\ \bar{h}^m_i \bar{h}^k_j H_{mk} &= \frac{1}{2}\{trH - H(N,N)\}\bar{h}_{ij}. \end{aligned} \tag{3.18}$$

Then, from (3.17)-(3.18) and Propostion 2, we have $R_{0ijk;0} = 0$ if and only if

$$H_{il} N^i \bar{h}^l_m = 0 \quad \text{and} \quad K_{ij} = \left(\frac{trK}{2}\right)\bar{h}_{ij}, \tag{3.19}$$

where $trK \equiv h^{ij} K_{ij} = \left\{trH - H_{ij} N^i N^j\right\}/||\widetilde{df}||$. The *first equation* in (3.19) implies that $||\widetilde{df}||$ is a function of f: this follows from the equality, $H_{il} N^i \bar{h}^l_m = (1/2||\widetilde{df}||)\left(||\widetilde{df}||^2\right)_{||m}$, where '$||$' denotes covariant derivative on $({}^{(2)}S,\ \bar{h})$; this equality can be derived from

$$\left(h^{ij} f_i f_j\right)_{||m} = 2H_{il} f^i \bar{h}^l_m\ . \tag{3.20}$$

The *second equation* in (3.19) shows that the hypersurface ${}^{(2)}S \subset {}^{(3)}\Sigma$ in STE spacetimes is totally umbilic (and hence, locally orientable) if $R_{0ijk;0} = 0$. Next, we show that the mean curvature, (trK), depends only on f.

Lemma 1: The mean curvature, (trK), of the the hypersurface, $({}^{(2)}S,\ \bar{h}) \subset ({}^{(3)}\Sigma,\ h)$ in STE spacetimes is constant on ${}^{(2)}S$ if $R_{0ijk;0} = 0$.

Proof. From (3.19), we insert $K^l_i = \frac{1}{2}(trK)\bar{h}^l_i$ in the Codazzi equation [20] for the hypersurface, $({}^{(2)}S,\ \bar{h}) \subset ({}^{(3)}\Sigma,\ h)$:

$$\hat{R}_{mj} N^j \bar{h}^m_i = K^l_{i||l} - (trK)_{||i} = -\frac{1}{2}(trK)_{||i}\ . \tag{3.21}$$

From (2.13) and (3.18), we also have

$$\hat{R}_{mj}N^j\bar{h}_i^m = (\frac{1}{f}H_{mj} + \Lambda h_{mj})N^j\bar{h}_i^m = 0\ . \tag{3.22}$$

From (3.21)-(3.22), it follows that $(trK)_{||i} = 0$.

We recall that the scalar curvature, $\bar{R}$, of the $f = \textit{constant}$ hypersurface, $(\,^{(2)}S,\ \bar{h}) \subset (\,^{(3)}\Sigma,\ h)$, is given by the Gauss equation [20],

$$\bar{R} = \hat{R} - 2\hat{R}_{ij}N^iN^j + (trK)^2 - trK^2. \tag{3.23}$$

Using (3.19) as well as the relations $\hat{R} = 2\Lambda$ and $\hat{R}_{ij}N^iN^j = \frac{1}{f}H_{ij}N^iN^j + \Lambda$ - derived from the equations (2.13)-(2.14), we find

$$\bar{R} = \frac{2}{f}\left\{||\widetilde{df}||(trK) + \Lambda f\right\} + \frac{1}{2}(trK)^2. \tag{3.24}$$

Since $||\widetilde{df}||$ is a function of f by (3.19)-(3.20), it follows from the Lemma 1 that $\bar{R}(f)$ is constant on the level surfaces, $^{(2)}S$, of f. Thus, in a STE spacetime if $R_{0ijk;0} = 0$ or equivalently, $(\,^{(3)}\Sigma, h)$ is conformally flat, then $^{(3)}\Sigma$ is locally foliated by totally umbilic [(3.19)] surfaces [$^{(2)}S$] of constant curvature [(3.24)], and hence, $(\,^{(3)}\Sigma, h)$ is locally a warp-product space, $\mathbb{R}\times_{r(f)}{}^{(2)}S$, for some function $r(f)$:

$$h = \frac{1}{||\widetilde{df}||^2}df\otimes df + r(f)^2\sigma_k, \tag{3.25}$$

where σ_k is the metric of constant curvature $k = 0, \pm 1$ on $^{(2)}S$.

3.4 Non-trivial static solutions to $R_{\mu\nu} = \Lambda g_{\mu\nu}$

If $(\,^{(3)}\Sigma, h)$ is conformally flat, then the general form of the static solutions to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ is given by

$$g = -f^2dx^0\otimes dx^0 + u(f)^2df\otimes df + r(f)^2\sigma_k, \tag{3.26}$$

where we have used (1.2) as well as (3.25), and set $u(f) \equiv 1/||\widetilde{df}||$. The functions $r(f)$ and $u(f)$ must satisfy the field equations [(2.13)-(2.14)]

$$\frac{r''}{r} - \frac{r'}{r}\frac{u'}{u} = -\frac{1}{2}\Lambda u^2 + \frac{1}{2f}\frac{u'}{u}, \tag{3.27}$$

$$2\frac{r'}{r} - \frac{u'}{u} = -\Lambda f u^2, \tag{3.28}$$

where a *prime* denotes differentiation with respect to f.

Assuming $r(f)$ is invertible, we set $\psi(r) = (df/dr)u(f(r))$ in (3.26), and the solution to (3.27)-(3.28) is given by $\psi(r) = 1/f(r)$ where $f^2(r) = k - 2\mu/r - (\Lambda/3)r^2$. This leads to spacetimes in Kottler class,

$$g_{kot} = -\left(k - \frac{2\mu}{r} - \frac{\Lambda}{3}r^2\right)dx^0\otimes dx^0 + \left(k - \frac{2\mu}{r} - \frac{\Lambda}{3}r^2\right)^{-1}dr\otimes dr + r^2\sigma_k,\quad \mu\in\mathbb{R}, \tag{3.29}$$

If $r(f) = c$ is a *nonzero constant*, it follows from (3.26) and $R_{\mu\nu} = \Lambda g_{\mu\nu}$ that $k/c^2 = \Lambda$. For $k \neq 0$ (or equivalently, $\Lambda \neq 0$), one obtains spacetimes in Nariai class [8]),

$$g_{nar} = -\left(a - \Lambda\rho^2\right) dx^0 \otimes dx^0 + \left(a - \Lambda\rho^2\right)^{-1} d\rho \otimes d\rho + \frac{1}{|\Lambda|}\sigma_k, \quad k = \frac{\Lambda}{|\Lambda|}, \quad a \in \mathbb{R}, \qquad (3.30)$$

where $\rho \equiv ||\widetilde{df}||/|\Lambda|$ and $f^2 = a - \Lambda\rho^2$.

Finally, we consider two remaining possible special cases of (3.26). (i) $r(f) = c$ and $k = 0$ which yields $\Lambda = 0$, and (ii) $u(f)$ is a constant which implies $r(f)$ is a constant and hence, $\Lambda = 0$. Each of these cases leads to a flat metric. Thus, conformal flatness of $({}^{(3)}\Sigma, h)$ implies that any non-trivial static solution solution to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ must be of Kottler-type [(3.29)] or Nariai-type [(3.30)]. This completes the proof of the Main Theorem.

4. Conclusions

We conclude with some remarks on possible generalizations of the Main Theorem.

(a) The relation, $R_{0ijk;0} = (-f^2)\hat{R}_{ijk}$, in (2.16) holds for an n-dimensional static spacetime, ${}^{(n-1)}\Sigma \times_f \mathbb{R}$, which is also an Einstein space. Here, the Cotton tensor of ${}^{(n-1)}\Sigma$ is given by

$$\hat{R}_{ijk} = (\hat{R}_{ij;k} - \hat{R}_{ik;j}) - \frac{1}{2(n-2)}(h_{ij}\hat{R}_{;k} - h_{ik}\hat{R}_{;j}).$$

(b) For $n > 4$, the Cotton tensor of ${}^{(n-1)}\Sigma$ vanishes if and only if its Weyl curvature is divergence free [6]:

$$\hat{C}^m{}_{ijk;m} = -\frac{(n-4)}{(n-3)}\hat{R}_{ijk}.$$

Thus, the conclusion of part-(3) of the Main Theorem holds for an n-dimensional STE spacetime if ${}^{(n-1)}\Sigma$ is conformally flat for $n > 4$. An example of this case is the $Schwarzschild - AdS_5$ static solution (see Appendix of [22]) to the 5-dimensional Einstein equations with a cosmological constant. The recent proof of the *Riemannian Penrose inequality* [23] has established the lower bound for the mass of an asymptotically flat black hole spacetime. It would be interesting to see if and how this rigorous result can be extended to n-dimensional ($n \geq 4$) toplogical black holes which are asymptotically locally AdS.

(c) For a stationary spacetime $(M^4,\ g)$ there is a coordinate system $\{x^0,\ x^1,\ x^2,\ x^3\}$ at each point of M^4 with $\partial_0 g_{\mu\nu} = 0$, and

$$g = -f^2(dx^0 + \omega) \otimes (dx^0 + \omega) + h_{ij}dx^i \otimes dx^j,$$

where $g_{00} = -f^2$, $\omega \equiv \omega_i dx^i$, $\omega_i \equiv g_{0i}/g_{00}$, and $h_{ij} \equiv g_{ij} - g_{00}\omega_i\omega_j$, for $i,\ j = 1,\ 2,\ 3$ and $\mu,\ \nu = 0, 1,\ 2,\ 3$. The 2-tensor, h_{ij}, is the metric on the space of integral curves [24] of the timelike Killing vector field (∂_0), and $R_{0ijk;0}$ defines a rank-3 tensor on this space. In this case, the structure of $R_{0ijk;0}$ is more complicated although the equation (2.16) holds in the static limit ($\omega = 0$). For the Kerr spacetime [20], one can check that $R_{0ijk;0}$ vanishes. Thus, it may be worthwhile to examine the implications of $R_{0ijk;0} = 0$ for the local geometry of stationary spacetimes which are Einstein spaces.

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